Replacing a Theoretical Probability by an Empirical Estimate

Project: Quantum clock synchronization (QCS)

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The highlighted phrase,

> "Replacing P(0) by the empirical estimate widehatP(0),"

marks a small but important transition in the parent document's reasoning. Up to this point, the argument has been written in terms of an ideal probability P(0): the probability that a quantum measurement produces outcome 0. But an experimenter never directly observes P(0). They observe a finite sequence of outcomes, count how often 0 appears, and use that count to estimate the probability.

So the highlighted passage is the moment where the parent document moves from the theoretical measurement model to a data-based estimator. It is not merely a notational substitution. It changes the status of the equation from an exact relationship between ideal quantities into an approximate inference from random experimental data.

The Theoretical Quantity P(0)

In the parent document, the ideal Ramsey-like measurement model is

> P(0)=(1+cos□□)/(2),

where P(0) is the true probability of observing outcome 0, and □□ is the quantum phase associated with the clock offset. This formula is directly supported by the parent document's earlier derivation: a relative phase in a two-level quantum state changes the interference pattern, and the Born rule turns that interference into a probability [Nielsen and Chuang 2010]. Ramsey-type experiments use precisely this kind of phase-to-probability conversion [Ramsey 1950].

If P(0) were known exactly, then one could algebraically invert the formula:

> cos□□ = 2P(0)-1.

From there, at least in the idealized case, one might write

> □□ = arccosbigl(2P(0)-1bigr).

This is still a theoretical relation. It says: if the true measurement probability were known, then the phase could be inferred, subject to the usual ambiguities of the cosine function. But the true probability is not observed directly. It is a parameter of the statistical model.

That is why the highlighted passage appears.

The Empirical Estimate widehatP(0)

Suppose the measurement is repeated N times under nominally identical conditions. Let n□ be the number of trials in which outcome 0 is observed. The empirical estimate of the probability is then

> widehatP(0)=(n□)/(N).

The hat over P is significant. It indicates that widehatP(0) is an estimator: a quantity computed from data. Before the experiment is performed, widehatP(0) is random, because n□ is random. After the experiment, it becomes a particular numerical value.

The statistical assumption usually sitting behind this step is binomial sampling. If each trial is independent, prepared in the same way, and measured with the same true probability P(0), then

$$n_0 \sim \mathrm{Binomial}\bigl(N, P(0)\bigr).$$

This means

$$\Pr(n_0 = k) = \binom{N}{k} P(0)^k \bigl(1-P(0)\bigr)^{N-k},$$

where k is a possible number of observed 0 outcomes. Under this model, $\widehat{P}(0)=n_0/N$ is an unbiased estimator of $P(0)$:

$$\mathbb{E}\left[\widehat{P}(0)\right]=P(0),$$

and its variance is

$$\mathrm{Var}\left(\widehat{P}(0)\right) = \frac{P(0)\bigl(1-P(0)\bigr)}{N}.$$

Thus, replacing $P(0)$ by $\widehat{P}(0)$ is justified as a standard statistical approximation when the number of trials is large and the trials are well modeled as independent repetitions [Casella and Berger 2002; Wasserman 2004].

The Plug-In Step

The highlighted phrase describes what statisticians often call a plug-in estimate. One starts with a formula involving an unknown parameter and