\[U(t)=e^{-iHt/\hbar}. \]

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The Time-Evolution Operator Behind a Rotating Quantum Phase

References

[Sakurai and Napolitano 2017] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, 2nd ed., Cambridge University Press, 2017.

[Nielsen and Chuang 2010] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, 10th Anniversary ed., Cambridge University Press, 2010.

[Stone 1932] M. H. Stone, "On One-Parameter Unitary Groups in Hilbert Space," Annals of Mathematics 33, no. 3, 643–648 (1932). DOI: 10.2307/1968538

[Ramsey 1950] N. F. Ramsey, "A Molecular Beam Resonance Method with Separated Oscillating Fields," Physical Review 78, 695–699 (1950). DOI: 10.1103/PhysRev.78.695

The highlighted formula

$$U(t)=e^{-iHt/\hbar}$$

is the operator-level version of the phase factor $e^{-i\omega t}$ discussed in the parent document. The parent document directly supports the claim that the minus sign in $e^{-i\omega t}$ comes from Schrödinger time evolution. The highlighted passage is that rule in its compact general form: instead of saying that a single energy component acquires the scalar phase $e^{-iEt/\hbar}$, it says that an entire quantum state evolves by applying the operator $U(t)$.

Here $U(t)$ is called the time-evolution operator or propagator. It takes a state at time 0, written $|\psi(0)\rangle$, to a state at time t, written

$$|\psi(t)\rangle=U(t)|\psi(0)\rangle$$

The Hamiltonian H is the operator representing the system's energy, $\hbar$ is the reduced Planck constant, and i is the imaginary unit. The exponential is not merely an ordinary scalar exponential unless H has already been replaced by one of its eigenvalues. It is an operator exponential, meaning that the same exponential idea is being applied to a matrix or, more generally, to a linear operator.

The formula is standard for a closed quantum system with a time-independent Hamiltonian [Sakurai and Napolitano 2017]. It is the precise mathematical bridge between "energy" and "phase rotation."

From Schrödinger's Equation to the Exponential

The starting point is the time-dependent Schrödinger equation,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle=H|\psi(t)\rangle$$

This equation says that the instantaneous rate of change of the quantum state is generated by the Hamiltonian. If H is independent of time, the equation resembles an ordinary first-order differential equation of the form

$$\frac{d}{dt}x(t)=Ax(t),$$

whose solution is $x(t)=e^{At}x(0)$. In the quantum case, rearranging Schrödinger's equation gives

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle$$

So the formal solution is

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

This is the highlighted expression. The object multiplying the initial state is defined as

$$U(t)=e^{-iHt/\hbar}$$

The minus sign and the factor of i are not decorative. They are exactly what make the evolution oscillatory rather than exponentially growing or decaying. If H has real energy eigenvalues, then $e^{-iEt/\hbar}$ has magnitude one. Thus a closed system's state vector changes phase while preserving total probability.

That preservation is encoded in the fact that U(t) is unitary:

$$U^\dagger(t)U(t)=I,$$

where $U^\dagger(t)$ is the adjoint of U(t), and I is the identity operator. Physically, unitarity means that inner products, norms, and therefore total probabilities are conserved. Mathematically, this rests on the Hamiltonian being self-adjoint, the operator-theoretic condition corresponding to observable real energies. Stone's theorem gives the deeper result that strongly continuous one-parameter unitary time evolutions are generated by self-adjoint operators [Stone 1932]. For most qubit calculations, this functional-analytic background is hidden, but it is part of what makes the formula more than notation.

What an Operator Exponential Means

For a finite-dimensional system, such as an ideal qubit, the operator exponential can be understood through its power series:

$$e^{-iHt/\hbar} = I-\frac{iHt}{\hbar} +\frac{1}{2!}\left(-\frac{iHt}{\hbar}\right)^2 +\frac{1}{3!}\left(-\frac{iHt}{\hbar}\right)^3+\cdots$$

Here H^2, H^3, and so on mean repeated matrix multiplication. This series definition is often the most direct way to see that the exponential of an operator is itself an operator.

There is an even clearer interpretation when H has energy eigenstates. Suppose

$$H|E_n\rangle=E_n|E_n\rangle.$$

Then applying U(t) to one such eigenstate gives

$$U(t)|E_n\rangle = e^{-iHt/\hbar}|E_n\rangle = e^{-iE_nt/\hbar}|E_n\rangle.$$

So the operator exponential reduces to an ordinary complex phase on each definite-energy component. This is exactly the mechanism used in the parent document. The scalar factor $e^{-i\omega t}$ is not separate from the highlighted formula; it is what the highlighted formula becomes after comparing two different energy eigenvalues.

If an initial state is a superposition,

$$|\psi(0)\rangle=\sum_n c_n|E_n\rangle,$$

then linearity gives

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle.$$

Each energy component carries its own rotating phase. The observable content usually lies not in any single phase by itself, but in the relative phases between components.

Recovering the Parent Document's Qubit Phase

The parent document considers a two-level system, with basis states $|0\rangle$ and $|1\rangle$, and explains the state

$$|\psi(t)\rangle = \frac{|0\rangle+e^{-i\omega t}|1\rangle}{\sqrt{2}}.$$

The highlighted formula explains where that phase comes from. If $|0\rangle$ and $|1\rangle$ are energy eigenstates,

$$H|0\rangle=E_0|0\rangle,\ H|1\rangle=E_1|1\rangle,$$

then

$$U(t)|0\rangle=e^{-iE_0t/\hbar}|0\rangle,\ U(t)|1\rangle=e^{-iE_1t/\hbar}|1\rangle.$$

Starting from

$$|\psi(0)\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}},$$

time evolution gives

$$|\psi(t)\rangle = \frac{e^{-iE_0t/\hbar}|0\rangle + e^{-iE_1t/\hbar}|1\rangle}{\sqrt{2}}.$$

Factoring out $e^{-iE_0t/\hbar}$ yields

$$|\psi(t)\rangle = e^{-iE_0t/\hbar}\frac{|0\rangle + e^{-i(E_1-E_0)t/\hbar}|1\rangle}{\sqrt{2}}.$$

The leading factor is a global phase, common to the whole state. In ordinary quantum measurements, such a global phase has no direct observable effect [Sakurai and Napolitano 2017]. Removing it leaves the relative phase

$$e^{-i(E_1-E_0)t/\hbar}.$$

Thus the parent document's ω is identified as

$$\omega=\frac{E_1-E_0}{\hbar}.$$

This derivation is directly aligned with the parent document's explanation: the qubit "ticks" because the energy splitting causes a relative phase to advance at a predictable angular frequency.

Why the Formula Uses H, Not Just E

A useful way to read the highlighted expression is that H contains all the possible energy-dependent phase rotations at once. If the state happens to be a single energy eigenstate, H effectively acts like the number E, and the result is a single phase $e^{-iEt/\hbar}$. If the state is a superposition of energy eigenstates, H applies the appropriate phase to each component.

This is why the operator form is essential. A qubit Hamiltonian might be written, for example, as

$$H=\frac{\hbar\omega}{2}\sigma_z,$$

where σ_z is the Pauli z-matrix. Then

$$U(t)=e^{-i\omega t\sigma_z/2}.$$

On the Bloch sphere, this describes a rotation about the z-axis. Depending on the convention for which state has which σ_z eigenvalue, the relative phase may appear as $e^{-i\omega t}$ or $e^{+i\omega t}$. This supports the parent document's warning that the sign in the exponent should be checked against the Hamiltonian and basis conventions rather than interpreted in isolation. Quantum computation texts often present single-qubit phase evolution in exactly this operator language [Nielsen and Chuang 2010].

When the Formula Needs Modification

The highlighted expression is exact under an important condition: the Hamiltonian must be time-independent. If the Hamiltonian changes with time, Schrödinger evolution is still unitary for a closed system, but the simple exponential generally becomes insufficient. One writes instead

U(t) = mathcalTexp□ft(-(i)/(hbar)□□□ H(t') dt'),

where mathcalT is the time-ordering operator. Time ordering is needed because Hamiltonians at different times may fail to commute:

[H(t□),H(t□)]□0.

If they do commute for all relevant times, the time-ordered expression can reduce to an ordinary exponential of the integral. But without that commutation, treating H(t) as if it were a constant can give the wrong evolution.

There are other boundaries to keep in mind. The formula describes closed-system unitary evolution. If the qubit interacts significantly with an uncontrolled environment, its state may decohere, and a density-matrix description with non-unitary effective dynamics may be required. The phase e□-i□ t□ may then lose visibility even if the underlying Hamiltonian still exists. This matters for any "ticking qubit" interpretation: the clock-like phase is useful only while coherence and a suitable phase reference are maintained. Ramsey interferometry is a standard operational setting in which such accumulated phases are converted into measurable population differences [Ramsey 1950].

What the Highlighted Passage Helps Verify

The passage

$$U(t)=e^{-iHt/\hbar}$$

is therefore the parent document's hidden engine. It supports the transition from energy splitting to phase evolution, and it explains why the phase factor has unit magnitude, why the exponent contains a minus sign under the usual Schrödinger convention, and why the frequency is naturally $(E_1-E_0)/\hbar$. But it does not, by itself, verify that a particular qubit is a good clock. To check that application, one must confirm that the Hamiltonian is effectively time-independent or correctly time-ordered, that |0⟩ and |1⟩ are being treated in the right energy basis, that the global phase has been removed consistently, and that the relative phase can actually be measured against an appropriate reference before decoherence erases it.